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# Block-Diagonal Equations for Multibody Elastodynamics with Geometric Stiffness and Constraints

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**This paper presents a comprehensive, block-diagonal matrix formulation of the equations of motion of a system of hinge-connected flexible bodies undergoing large rotation and translation together with small elastic vibration. The formulation compensates for premature linearization of equations, associated with the customary treatment of small elastic displacement, by accounting for geometric stiffness due to inertia as well as interbody forces. The algorithm is first developed for a tree configuration and is then extended to the case of closed structural loops by cutting the loops and expressing all of the kinematical variables into terms dependent and free of constraint forces/torques. A solution procedure satisfying constraints is given.**

## Introduction

MULTIBODY dynamics has come a long way since the pioneering work of Hooker and Margulies[1] published about 25 years ago. When the bodies can be treated as rigid, the theory is exact, and books, e.g., Refs. 2 and 3, and efficient computer programs[4,5] dealing with them are now available. When the bodies in the system must be considered as deformable, useful theories are only approximate. Typical multi-flexible-body dynamics codes[6,7] are based on the assumption of small elastic displacement of a body in a reference frame that can go through large rotation and translation; the elastic displacement is expressed by superposition of assumed modes and dynamical equations are linearized in the modal coordinates. However, this process is unavoidably one of premature linearization and, although producing acceptable simulations in many benign situations, can yield completely erroneous results[8,9] in more stringent applications. Correct linearization requires taking care of the deformation constraints of the particular continuum, for example, beam or plate.[8,10] One can, however, compensate for premature linearization by the consideration of geometric stiffness[11–17] that is iteratively updated with deformation. Recently, an approximate theory[18] of motion-induced stiffness of arbitrary structures has been developed that does not require repeated forming of the geometric stiffness matrix, and this method has been implemented in a multibody dynamics formalism.[19]

Formulations of the equations of motion of a multibody system in all of the previous references give rise to a high-order, dense "mass" matrix. It can be shown that the procedure used to form the coefficients of the mass matrix and then to uncouple the equations with respect to the highest time derivative of the dependent variables require $\mathcal{O}(n^3)$ arithmetic operations, where $n$ is the number of degrees of freedom of the system. Recursive or order-$n$ formulations, where the computational count varies as $n$, has been a subject of intense research in the field of multibody dynamics recently.[3,20–31] There is a basic similarity in the $\mathcal{O}(n)$ formulations published so far in that all use recursive methods in three passes: a forward pass to construct the kinematics, a backward pass to get the first dynamical equation, and a final forward pass to get the rest of the dynamical equations.

The method behind order-$n$ formulations, however, varies. For example, Ref. 20 uses Lagrange's equations whereas Refs. 21, 22, and 31 use Kane's equations; Refs. 23 and 24 use an operator algebra technique in deriving the equations; Refs. 25 and 26 use the method of virtual work; Refs. 3 and 27 use the Newton-Euler method, and Ref. 28 uses a velocity transformation approach. References 29 and 30 apply the procedure behind deriving order-$n$ equations to a system of flexible bodies. However, Ref. 29 introduces many Lagrange multipliers where none are needed with a different formulation strategy, and Ref. 30 does not treat constraints. None of these papers include considerations for motion-induced stiffness.

The contribution of the present paper is twofold. First, it gives a block-diagonal formulation of the equations of multi-flexible-body dynamics accounting for geometric stiffness due to inertia as well as interbody forces,[32] a feature required for correct representation of flexible body dynamics over a wide range of motion. Second, the paper gives a detailed treatment of the equations of motion of systems with holonomic constraints, in a manner that preserves the block-diagonal formulation and satisfies the constraints at the position, velocity, and acceleration levels. The following section gives a simple example illustrating how geometric stiffness due to inertia and interbody forces compensates for premature linearization in a multibody context. The main text then describes a multibody system and develops the recursive kinematics, the block-diagonal algorithm incorporating geometric stiffness, and the treatment for constraints.

## Example

The example described here extends the simple example of Ref. 18 to establish the need for considering geometric stiffness due to interbody forces as well as inertia forces when two

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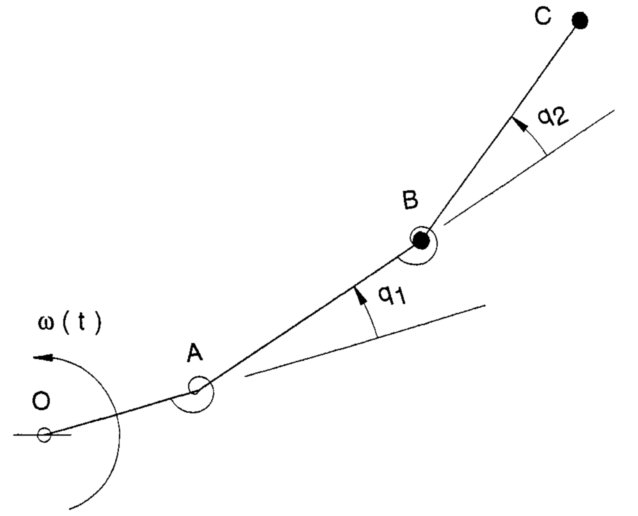


**Fig. 1 Simple system having geometric stiffness due to interbody and inertia forces.**

or more flexible bodies are attached in series. Figure 1 shows three massless rigid rods, $OA$, $AB$, $BC$, connected at the hinges $A$ and $B$ by torsional springs of stiffness $k$. Rod $OA$ of length $r$ is driven by a motor at the angular speed $\omega(t)$. Rods $AB$ and $BC$ are each of length $L$. A particle of mass $m$ is attached to $AB$ at $B$, and another particle of mass $m$ is attached to $BC$ at $C$. The angle between $OA$ and $AB$ is $q_1$, and the angle between $AB$ and $BC$ is $q_2$. The correct dynamical equations of this system for assumed small motion in $q_1$ and $q_2$ are

$$M\ddot{Q} + KQ = F, \qquad Q = [q_1, q_1]^T \tag{1}$$

where

$$M = mL^2\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}, \qquad F = -m\dot{\omega}L\begin{Bmatrix} 5L+3r \\ 2L+r \end{Bmatrix}$$

$$K = \begin{bmatrix} k+3m\omega^2 rL & m\omega^2 rL \\ m\omega^2 rL & k+m\omega^2 L(r+L) \end{bmatrix} \tag{2}$$

It is well known that, for correct linearization, nonlinear terms in the velocity expressions have to be kept up to a certain stage in the analysis. Thus, for example, if one linearizes the velocities of $B$ and $C$ in $q_1$ and $q_2$, and uses these in Lagrange's equations, one obtains the incorrect linear equations in the form of Eq. (1), with $M$ and $F$ remaining as in Eq. (2) but with the (wrong) stiffness matrix $K$,

$$K = \begin{bmatrix} k-5m\omega^2 L^2 & -2m\omega^2 L^2 \\ -2m\omega^2 L^2 & k-m\omega^2 L^2 \end{bmatrix} \tag{3}$$

It is apparent that the consequence of premature linearization in this case has been a loss of stiffness. If one is to use prematurely linearized equations, as one is forced to do in the case of arbitrary elastic bodies with the customary assumption of small deformation, what mechanism does exist for compensating for this loss of stiffness? The answer, as was given in Ref. 18, is geometric stiffness due to existing loads. In the case of a terminal body the existing forces include all of the inertia forces on the body. In the case of an intermediate body, the interbody forces may be as important[32] as the inertia forces on the body, and thus their combined contributions to geometric stiffness must be considered. For this particular example, inertia force magnitude on rod $BC$ in the undeformed configuration is

$$F^{BC} = m\omega^2(r+2L) \tag{4}$$

and the force on rod $AB$ consists of the inertia force for the mass on $AB$ and the interbody force applied by rod $BC$ on $AB$, which, in the absence of any external force on $BC$, is the inertia force of $BC$, thus yielding the interbody force magnitude,

$$F^{AB} = m\omega^2(r+L) + m\omega^2(r+2L) \tag{5}$$

Using the well-known form of the geometric stiffness matrix for a bar[33] and the displacements at the ends of the rods $AB$ and $BC$, one can write the potential energy associated with this load-dependent stiffness as

$$P_g = 0.5\frac{F^{AB}}{L}[0 \quad Lq_1]\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ Lq_1 \end{Bmatrix} + 0.5\frac{F^{BC}}{L}$$
$$\times\left[Lq_1 \quad L(2q_1+q_2)\right]\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} Lq_1 \\ L(2q_1+q_2) \end{Bmatrix} \tag{6}$$

Generalized active force due to this additional potential energy is

$$-\begin{Bmatrix} \dfrac{\partial P_g}{\partial q_1} \\ \dfrac{\partial P_g}{\partial q_2} \end{Bmatrix} = -[K_g]\{Q\}$$

$$[K_g] = m\omega^2 L\begin{bmatrix} 3r+5L & r+2L \\ r+2L & r+2L \end{bmatrix} \tag{7}$$

As can be checked, the matrix $K_g$ in Eq. (7) when added to the incorrect stiffness matrix of Eq. (3) gives precisely the correct stiffness matrix in Eq. (2). *Note also that the correct equations would not be obtained if the force $F^{AB}$ on AB accounted for only the inertia of B and not the interbody force from BC.* In other words, the error of premature linearization can only be compensated for by the addition of geometric stiffness effects associated with both inertia and interbody forces. This idea will be used subsequently in the context of multibody dynamics.

## Multibody System

A system of rigid and flexible bodies connected in a topological tree is shown in Fig. 2a. The bodies are numbered arbitrarily, with an inertial frame denoted as body 0 and a topological array is defined such that body $j$ has inboard adjacent body $c(j)$ in the path going from body $j$ to body 0. Figure 2b shows two adjacent bodies $B_j$ and $B_{c(j)}$ connected at a hinge allowing rotation and translation, with $Q_j$ a hinge point on $B_j$ and $P_j$ the corresponding hinge point on $B_{c(j)}$. Reference frames $j$ and $p_j$ are attached to $Q_j$ and $P_j$, respectively. Let $\tau^j$ denote a $(T_j \times 1)$ matrix of generalized coordinates representing $T_j$ relative translations of $Q_j$ relative to $P_j$; let $\theta^j$ denote a $(R_j \times 1)$ matrix of generalized coordinates representing $R_j$ relative rotations of frame $j$ with respect to $p_j$, and let $\eta^j$ denote a $(M_j \times 1)$ matrix of modal coordinates for body $j$. Note that this defines $n$ degrees of freedom of the $N$-body system, where $n$ is given by

$$n = \sum_{j=1}^{N}(R_j + T_j + M_j) \tag{8}$$

## Recursive Equations of Kinematics

In developing the kinematics, the basic assumption made regarding elastic displacements and rotations is that they are small and are given by modal superposition. Referring to Fig. 2b, the angular velocity of frame $j$ can be computed from the angular velocity of frame $c(j)$ as follows:

$$\omega^j = C^T_{c(j),j}\left[\omega^{c(j)} + \varphi^{c(j)}(P_j)\dot{\eta}^{c(j)} + C_{c(j),p(j)}G^j\dot{\theta}^j\right] \tag{9}$$

Here $C_{c(j),j}$ is the matrix transforming the $j$ frame to the $c(j)$ frame; $\varphi^{c(j)}(P_j)$ denotes the modal rotation at $P_j$; and $G^j$ is

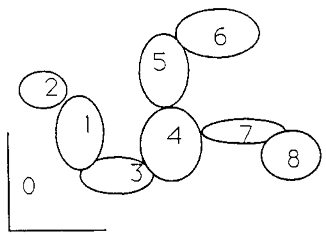


**Fig. 2a System of rigid and flexible bodies in a topological tree.**

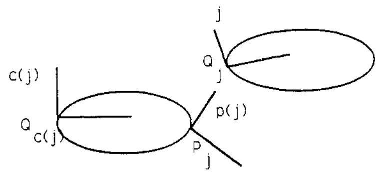


**Fig. 2b Two adjacent bodies connected at a hinge allowing relative rotation and translation.**

the gimbal transformation matrix between frames $P_j$ and $j$. Velocity of the point $Q_j$ in Fig. 2b can be obtained recursively as follows:

$$v^{Q_j} = C^T_{c(j),j}\Big\{v^{Q_{c(j)}} + \tilde{\omega}^{c(j)}\big[r^{Q_{c(j)}P_j} + \phi^{c(j)}(P_j)\eta^{c(j)}\big] + \phi^{c(j)}(P_j)\dot{\eta}^{c(j)} + C_{c(j),p(j)}L^j\dot{\tau}^j + \big[\tilde{\omega}^{c(j)} + \widetilde{\varphi^{c(j)}(P_j)\dot{\eta}^{c(j)}}\big]C_{c(j),p(j)}L^j\tau^j\Big\} \tag{10}$$

Here $L^j\tau^j$ denotes the $p(j)$-axis measure number of relative translational displacement of hinge $Q_j$, and the quantities with a tilde denote the usual the skew-symmetric matrix formed out of a $3\times 1$ matrix. Now, obtain the partial velocity[2] of $Q_j$ with respect to the relative translation at hinge $Q_j$ and the partial angular velocity[2] of frame $j$ with respect to the relative rotation of $j$ at hinge $Q_j$ by inspection from Eqs. (9) and (10) and use these to define the partial velocity/partial angular velocity matrix $R^j$

$$R^j = \begin{bmatrix} C^T_{p,j}L^j & 0 \\ 0 & C^T_{p,j}G^j \end{bmatrix} \tag{11}$$

Note that the use of Eqs. (9) and (10), which are linear in the modal coordinates, before the extraction of partial velocity/ partial angular velocity constitutes premature linearization.[8,10]

Now, following Ref. 21, separate the acceleration of $Q_j$ and the angular acceleration of frame $j$ into two groups of terms, one (associated subsequently with the subscript 0) that involves second derivatives of the generalized coordinates and another that does not.

$$a^{Q_j} = a_0^{Q_j} + a_t^{Q_j} \qquad \alpha^j = \alpha_0^j + \alpha_t^j \tag{12}$$

Further, separate $a_0^{Q_j}$ and $\alpha_0^j$ into a group representing all inboard body contributions and another associated with the hinge $Q_j$, so that

$$\begin{Bmatrix} a_0^{Q_j} \\ \alpha_0^j \end{Bmatrix} = \begin{Bmatrix} \hat{a}_0^{Q_j} \\ \hat{\alpha}_0^j \end{Bmatrix} + R^j\begin{Bmatrix} \ddot{\tau}^j \\ \ddot{\theta}^j \end{Bmatrix} \tag{13}$$

Differentiation of Eq. (9) and use of the preceding definitions lead to the recursive construction of $\alpha_t^j$

$$\alpha_t^j = C^T_{c(j),j}\Big\{\alpha_t^{c(j)} + \tilde{\omega}^{c(j)}\big[\varphi^{c(j)}(P_j)\dot{\eta}^{c(j)} + C_{c(j),p(j)}G^j\dot{\theta}^j\big] + C_{c(j),p(j)}\dot{G}^j\dot{\theta}^j + \big[\widetilde{\varphi^{c(j)}(P_j)\dot{\eta}^{c(j)}}\big]C_{c(j),p(j)}G^j\dot{\theta}^j\Big\} \tag{14}$$

Differentiation of Eq. (10) and use of the definition in Eq. (12) lead to the following recursive construction of $a_t^{Q_j}$

$$\begin{aligned} a_t^{Q_j} = C^T_{c(j),j}\Big\langle & a_t^{Q_{c(j)}} + \widetilde{\alpha_t^j}\big[r^{Q_{c(j)}P_j} + \phi^{c(j)}(P_j)\eta^{c(j)} + C_{c(j),p(j)}L^j\tau^j\big] + \tilde{\omega}^{c(j)}\big[\tilde{\omega}^{c(j)}(r^{Q_{c(j)}P_j} + \phi^{c(j)}(P_j)\eta^{c(j)}) \\ & + 2\phi^{c(j)}(P_j)\dot{\eta}^{c(j)}\big] + \big[\tilde{\omega}^{c(j)} + \widetilde{\varphi^{c(j)}(P_j)\dot{\eta}^{c(j)}}\big]\Big\{\big[\tilde{\omega}^{c(j)} + \widetilde{\varphi^{c(j)}(P_j)\dot{\eta}^{c(j)}}\big]C_{c(j),p(j)}L^j\tau^j + 2C_{c(j),p(j)}L^j\dot{\tau}^j\Big\} \\ & - \big[\widetilde{C_{c(j),p(j)}L^j\tau^j}\big]\tilde{\omega}^{c(j)}\varphi^{c(j)}(P_j)\dot{\eta}^{c(j)}\Big\rangle \end{aligned} \tag{15}$$

Equations (9), (14), and (15) are computed in a forward pass going from body 1 to body $N$ for use in a backward pass for generating the dynamical equations described in the next section.

## Block-Diagonal Equations of Dynamics

Equations of large overall motion of a single flexible body are well known and are the building blocks of the multibody equations. The emphasis here is on the exposition of the block-diagonal form of the multibody equations and, to save space, reference will be made to Refs. 18 and 19 for any part of the single-body equations not essential to our purpose. The modal equations for a terminal body $j$, due to inertia and external forces, thus can be written by reference to Eq. (10) of Ref. 19 in the matrix form,

$$E^j\ddot{\eta}^j + b^{j^T}a^{Q_j} + g^{j^T}\alpha^j - \Omega^j\Delta^j\omega^j + 2\Omega^j\rho^j\dot{\eta}^j + \Gamma_1^j a^{Q_j} + \Gamma_2^j\alpha^j + \Gamma_3^j + \lambda^j\eta^j = \int_j \phi_e^{j^T}\,\mathrm{d}f_e^j \tag{16}$$

where $E^j$ is the generalized modal mass; $b^j$ and $g^j$ are $(3\times M_j)$ modal integral matrices defined in Ref. 19; $\Omega^j$ is a $(M_j\times 3M_j)$ banded matrix with the body components of the angular velocity $\omega^j$ appearing in each row starting with the diagonal; $\Delta^j$ is $(3M_j\times 3)$ matrix made up of $3\times 3$ modal dyadics $D_i^j$ $(i=1,\ldots,M_j)$ defined in Ref. 19; and $\rho^j$ is a $(M_j\times M_j)$ modal integral defined in Ref. 7. Generalized structural stiffness is $\lambda^j$, and the right-hand side in Eq. (16) represents the generalized active force due to external forces on body $j$.

Geometric stiffness in Eq. (16) is described by the terms involving $\Gamma_i^j$ $(i=1, 2, 3)$. It has been shown in Ref. 18 that motion-induced stiffness of an arbitrary structure can be described by the geometric stiffness due to 12 inertia loads corresponding to the 12 columns of the $(3\times 12)$ load matrix given next for each node of the body,

$$\begin{Bmatrix} f_1^{*i} \\ f_2^{*i} \\ f_3^{*i} \end{Bmatrix} = -m_i[U_3 \quad x_iU_3 \quad y_iU_3 \quad z_iU_3]\begin{Bmatrix} a_1^j \\ \cdot \\ \cdot \\ a_{12}^j \end{Bmatrix} \tag{17}$$

where $U_3$ is the $3\times 3$ identity matrix, $m_i$ is the lumped mass at the $i$th node, $(x_i, y_i, z_i)$ are the coordinates of the $i$th node, and the 12 elements of the column matrix at the right refer to 12 terms in the expression for the acceleration of the hinge at the base of the structure. In terms of the notations of Eq. (12), these acceleration terms for the $j$th single body of a multibody system are

$$\begin{Bmatrix} a_1^j \\ a_2^j \\ a_3^j \end{Bmatrix} = \{a_0^{Q_j} + a_t^{Q_j}\}$$

$$\begin{Bmatrix} a_4^j \\ a_5^j \\ a_6^j \end{Bmatrix} = \begin{Bmatrix} z_1^j \\ z_2^j \\ z_3^j \end{Bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \{\alpha_0^j + \alpha_t^j\}$$

$$\begin{Bmatrix} a_7^j \\ a_8^j \\ a_9^j \end{Bmatrix} = \begin{Bmatrix} z_2^j \\ z_4^j \\ z_5^j \end{Bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \{\alpha_0^j + \alpha_t^j\} \tag{18}$$

$$\begin{Bmatrix} a_{10}^j \\ a_{11}^j \\ a_{12}^j \end{Bmatrix} = \begin{Bmatrix} z_3^j \\ z_5^j \\ z_6^j \end{Bmatrix} - \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \{\alpha_0^j + \alpha_t^j\}$$

where

$$\begin{bmatrix} z_1^j & z_2^j & z_3^j \\ z_2^j & z_4^j & z_5^j \\ z_3^j & z_5^j & z_6^j \end{bmatrix} = \tilde{\omega}^j \tilde{\omega}^j$$

Dynamic stiffness in Eq. (16) is now defined in terms of the 12 geometric stiffness matrices $K_{gi}^j$ due to unit values of $a_i^j$ ($i = 1, \dots, 12$) in Eq. (17) and the modal matrix $\phi^j$ and the modal coordinate $\eta^j$.

$$G_i^j = \phi^{j^T} K_{gi}^j \phi^j$$

$$\Gamma_1^j = [G_1^j \eta^j \quad G_2^j \eta^j \quad G_3^j \eta^j]$$

$$\Gamma_2^j = \left[(G_9^j - G_{11}^j)\eta^j \quad (G_{10}^j - G_6^j)\eta^j \quad (G_5^j - G_7^j)\eta^j\right]$$

$$\Gamma_3^j = G_4^j \eta^j z_1^j + G_5^j \eta^j z_2^j + G_6^j \eta^j z_3^j + G_7^j \eta^j z_2^j + G_8^j \eta^j z_4^j$$

$$+ G_9^j \eta^j z_5^j + G_{10}^j \eta^j z_3^j + G_{11}^j \eta^j z_5^j + G_{12}^j \eta^j z_6^j \tag{19}$$

Based on Eq. (12), Eq. (16) can be written as

$$E^j \ddot{\eta}^j = A^j \begin{Bmatrix} a_0^{Q_j} \\ \alpha_0^j \end{Bmatrix} + Y_1^j \tag{20}$$

where

$$A^j = -\left[b^{j^T} + \Gamma_1^j \quad g^{j^T} + \Gamma_2^j\right] \tag{21}$$

$$Y_1^j = -\left\{b^{j^T} a_t^{Q_j} + g^{j^T} \alpha_t^j - \Omega^j \Delta^j \omega^j + 2\Omega^j \rho^j \dot{\eta}^j + \Gamma_1^j a_t^{Q_j} + \Gamma_2^j \alpha_t^j + \Gamma_3^j + \lambda^j \eta^j\right\} + \int_j \phi_e^{j^T} \, \mathrm{d}f_e^j \tag{22}$$

The resultant inertia force and the moment of the inertia forces about the hinge $Q_j$, together with the external forces and moments, can be written following Eq. (10) of Ref. 19 as a set of equilibrated forces and moments in the matrix form,

$$\begin{Bmatrix} f^{*j} - f_e^j \\ t^{*j} - t_e^j \end{Bmatrix} = M_1^j \begin{Bmatrix} a_0^{Q_j} \\ \alpha_0^j \end{Bmatrix} + M_2^j \ddot{\eta}^j + X^j \tag{23}$$

where the following definitions are used in terms of mass $m^j$, and deformation-dependent first and second moments about the hinge, $s^{j/Q_j}$ and $I^{j/Q_j}$:

$$M_1^j = \begin{bmatrix} m^j & -\tilde{s}^{j/Q_j} \\ \tilde{s}^{j/Q_j} & I^{j/Q_j} \end{bmatrix} \tag{24}$$

$$M_2^j = \begin{bmatrix} b^j \\ g^j \end{bmatrix} \tag{25}$$

$$X^j = M_1^j \begin{Bmatrix} a_t^{Q_j} \\ \alpha_t^j \end{Bmatrix} + \begin{Bmatrix} \tilde{\omega}^j \tilde{\omega}^j s^{j/Q_j} + 2\tilde{\omega}^j b^j \dot{\eta}^j - F_e^j \\ \tilde{\omega}^j I^{j/Q_j} \omega^j + 2 \sum_{k=1}^{M_j} D_k^{j*} \dot{\eta}_k^j \omega^j - T_e^j \end{Bmatrix} \tag{26}$$

Here $D^{j*}$ is the transpose of $D^j$ defined in Ref. 19. Now using Eq. (20) in Eq. (23), and defining

$$M_3^j = M_1^j + M_2^j E^{j^{-1}} A^j$$

$$Y_2^j = X^j + M_2^j E^{j^{-1}} Y_1^j \tag{27}$$

allow Eq. (23) to be rewritten as

$$\begin{Bmatrix} f^{*j} - f_e^j \\ t^{*j} - t_e^j \end{Bmatrix} = M_3^j \begin{Bmatrix} a_0^{Q_j} \\ \alpha_0^j \end{Bmatrix} + Y_2^j \tag{28}$$

Kane's equations[2] associated with the hinge degrees of freedom for a terminal body $j$ are written by premultiplying the dynamic equilibrium equations, Eq. (28), by the transpose of the partial velocity/partial angular velocity matrix in Eq. (11), where multiplication with the zeros in Eq. (11) are avoided.

$$\begin{Bmatrix} \ddot{\tau}^j \\ \ddot{\theta}^j \end{Bmatrix} = -\nu^{j^{-1}} R^{j^T} \left[ M_3^j \begin{Bmatrix} \hat{a}_0^{Q_j} \\ \hat{\alpha}_0^j \end{Bmatrix} + Y_2^j \right] + \nu^{j^{-1}} \begin{pmatrix} f_h^j \\ t_h^j \end{pmatrix} \tag{29}$$

Here $f_h^j$ and $t_h^j$ are the hinge interaction force and torque on body $j$, and

$$\nu^j = R^{j^T} M_3^j R^j \tag{30}$$

Use of Eqs. (13) and (29) in Eq. (28), and the definitions

$$M = M_3^j - M_3^j R^j \nu^{j^{-1}} R^{j^T} M_3^j$$

$$X = Y_2^j - M_3^j R^j \nu^{j^{-1}} R^{j^T} Y_2^j + M_3^j R^j \nu^{j^{-1}} \begin{pmatrix} f_h^j \\ t_h^j \end{pmatrix} \tag{31}$$

lead to a re-expression of Eq. (28) as

$$\begin{Bmatrix} f^{*j} - f_e^j \\ t^{*j} - t_e^j \end{Bmatrix} = M \begin{Bmatrix} \hat{a}_0^{Q_j} \\ \hat{\alpha}_0^j \end{Bmatrix} + X \tag{32}$$

Now define the kinematical propagation equation

$$\begin{Bmatrix} \hat{a}_0^{Q_j} \\ \hat{\alpha}_0^j \end{Bmatrix} = W^j \begin{Bmatrix} a_0^{Q_{c(j)}} \\ \alpha_0^{c(j)} \end{Bmatrix} + N^j \ddot{\eta}^{c(j)} \tag{33}$$

where $W^j$ is a shift transform,

$$W^j = \begin{bmatrix} C_{c(j),j}^T & -C_{c(j),j}^T \tilde{r}_{c(j)}^{Q_{c(j)}Q_j} \\ 0 & C_{c(j),j}^T \end{bmatrix} \tag{34}$$

and the following additional notations have been introduced:

$$N^j = C^{j^T} \begin{Bmatrix} \phi^{c(j)}(P_j) - \overline{C_{c(j),p(j)} L^j \tau^j \varphi^{c(j)}(P_j)} \\ \varphi^{c(j)}(P_j) \end{Bmatrix} \tag{35}$$

$$C^j = \begin{bmatrix} C_{c(j),j} & 0 \\ 0 & C_{c(j),j} \end{bmatrix} \tag{36}$$

Here the skew-symmetric matrix in Eq. (34) is formed out of the position vector from $Q_{c(j)}$ to $Q_j$ resolved in the $c(j)$ basis,

$$r_{c(j)}^{Q_{c(j)}Q_j} = r^{Q_{c(j)}P_j} + \phi^{c(j)}(P_j)\eta^{c(j)} + C_{c(j),p(j)}L^j\tau^j \tag{37}$$

The equilibrated system of force and moment at $Q_j$, expressed in the $j$ basis, is obtained by substituting Eq. (33) in Eq. (32)

$$\begin{Bmatrix} f^{*j} - f_e^j \\ t^{*j} - t_e^j \end{Bmatrix}_{Q_j/j} = M\left(W^j \begin{Bmatrix} a_0^{Q_{c(j)}} \\ \alpha_0^{c(j)} \end{Bmatrix} + N^j\ddot{\eta}^{c(j)}\right) + X \tag{38}$$

The interbody force applied by body $j$ on body $c(j)$ at $P_j$ is obtained by replacing the force system of Eq. (38) from $Q_j$ to $P_j$. In the $c(j)$ basis this is expressed as

$$\begin{Bmatrix} f^{*j} - f_e^j \\ t^{*j} - t_e^j \end{Bmatrix}_{P_j/c(j)} = d_j^{c(j)}\left[M\left(W^j \begin{Bmatrix} a_0^{Q_{c(j)}} \\ \alpha_0^{c(j)} \end{Bmatrix} + N^j\ddot{\eta}^{c(j)}\right) + X\right] \tag{39}$$

where

$$d_j^{c(j)} = \begin{bmatrix} C_{c(j),j} & 0 \\ C_{c(j),j}\widetilde{C_{p,j}^T L^j\tau^j} & C_{c(j),j} \end{bmatrix} \tag{40}$$

Geometric stiffness caused by interbody forces on body $c(j)$ contributes to generalized active force, written keeping linear terms in $\eta^{c(j)}$, as follows, where $S_j^{c(j)}$ is a $6\times 6$ identity matrix except for a zero row corresponding to a rotation axis in the $p(j)$ frame for which no moment can be transferred:

$$F_{\text{interbody}}^{c(j)} = -\sum_{i=1}^{6} G_{12+i}^{c(j)}\eta^{c(j)}S_j^{c(j)}d_j^{c(j)}\left[MW^j \begin{Bmatrix} a_0^{Q_{c(j)}} \\ \alpha_0^{c(j)} \end{Bmatrix} + X\right] \tag{41}$$

Contribution to the equations of motion associated with $\eta^{c(j)}$ for an inner body come from three sources: the inertia and active forces on body $c(j)$, the geometric stiffness effects due to inertia *and* interbody forces, and the equilibrated system of forces and moments acting at $Q_j$. With the use of Eq. (12) this is written as

$$E^{c(j)}\ddot{\eta}^{c(j)} - A^{c(j)}\begin{Bmatrix} a_0^{Q_{c(j)}} \\ \alpha_0^{c(j)} \end{Bmatrix} - Y_1^{c(j)} - F_{\text{interbody}}^{c(j)} + N^{j^T}\begin{Bmatrix} f^{*j} - f_e^j \\ t^{*j} - t_e^j \end{Bmatrix}_{Q_j/j} = 0 \tag{42}$$

Substituting Eqs. (38) and (41) in Eq. (42) and collecting terms, one gets the modal equations for $\ddot{\eta}^{c(j)}$ in the same form as Eq. (20),

$$E^{c(j)}\ddot{\eta}^{c(j)} = A^{c(j)}\begin{Bmatrix} a_0^{Q_{c(j)}} \\ \alpha_0^{c(j)} \end{Bmatrix} + Y_1^{c(j)} \tag{43}$$

after the following replacements indicated by arrows have been made:

$$\begin{aligned} E^{c(j)} &\rightarrow E^{c(j)} + N^{j^T}MN^j \\ A^{c(j)} &\rightarrow A^{c(j)} - N^{j^T}MW^j - \sum_{i=1}^{6} G_{12+i}^{c(j)}\eta^{c(j)}S_i^{c(j)}d_i^{c(j)}MW^j \\ Y_1^{c(j)} &\rightarrow Y_1^{c(j)} - N^{j^T}X - \sum_{i=1}^{6} G_{12+i}^{c(j)}\eta^{c(j)}S_i^{c(j)}d_i^{c(j)}X \end{aligned} \tag{44}$$

Now, the forces and moments acting at $Q_j$ can be replaced at $Q_{c(j)}$ and expressed in the $c(j)$ basis by using the shift transform of Eq. (34),

$$\begin{Bmatrix} f^{*j} - f_e^j \\ t^{*j} - t_e^j \end{Bmatrix}_{Q_{c(j)}/c(j)} = W^{j^T}\begin{Bmatrix} f^{*j} - f_e^j \\ t^{*j} - t_e^j \end{Bmatrix}_{Q_j/j} \tag{45}$$

To this set of forces and moments is added the resultant of all of the active and inertia forces and moments about $Q_{c(j)}$, yielding

$$\sum_{j\in S_o}\begin{Bmatrix} f^{*j} - f_e^j \\ t^{*j} - t_e^j \end{Bmatrix} = M_1^{c(j)}\begin{Bmatrix} a_0^{Q_{c(j)}} \\ \alpha_0^{c(j)} \end{Bmatrix} + M_2^{c(j)}\ddot{\eta}^{c(j)} + X^{c(j)} \tag{46}$$

where $S_o$ is the set of all bodies outboard of the hinge $Q_{c(j)}$, and the following replacements have been made:

$$\begin{aligned} M_1^{c(j)} &\rightarrow M_1^{c(j)} + W^{j^T}MW^j \\ M_2^{c(j)} &\rightarrow M_2^{c(j)} + W^{j^T}MN^j \\ X^{c(j)} &\rightarrow X^{c(j)} + W^{j^T}X \end{aligned} \tag{47}$$

Now, the solution for $\ddot{\eta}^{c(j)}$ is used from Eq. (43) in Eq. (46), and Eq. (13) is invoked with $j$ replaced by $c(j)$, and Kane's dynamical equations associated with the degrees of freedom at the hinge $Q_{c(j)}$ are written as before.

$$\begin{Bmatrix} \ddot{\tau}^{c(j)} \\ \ddot{\theta}^{c(j)} \end{Bmatrix} = -\nu^{c(j)^{-1}}R^{c(j)^T}\left[M_3^{c(j)}\begin{Bmatrix} \hat{a}_0^{Q_{c(j)}} \\ \hat{\alpha}_0^{c(j)} \end{Bmatrix} + Y_2^{c(j)}\right] + \nu^{c(j)^{-1}}\begin{pmatrix} f_h^{c(j)} \\ t_h^{c(j)} \end{pmatrix} \tag{48}$$

This reproduces two cycles of the formulation of dynamical equations, covering the degrees of freedom of two bodies. This pattern is repeated until the equations for the base body (body 1) are written; at this stage, the inboard body accelerations/angular accelerations are zero or prescribed. Assuming that the zeroth body is the inertial frame, Eq. (48) reduces for $j=1$ to

$$\begin{Bmatrix} \ddot{\tau}^j \\ \ddot{\theta}^j \end{Bmatrix} = -\nu^{j^{-1}}R^{j^T}Y_2^j + \nu^{j^{-1}}\begin{pmatrix} f_h^j \\ t_h^j \end{pmatrix} \tag{49}$$

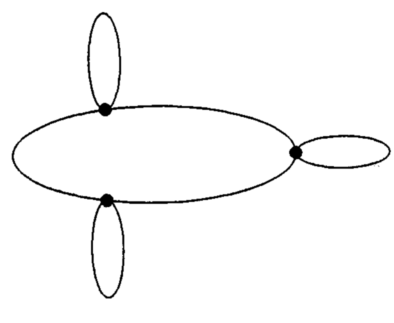

**Fig. 3a Rigid body with three articulated flexible appendages.**

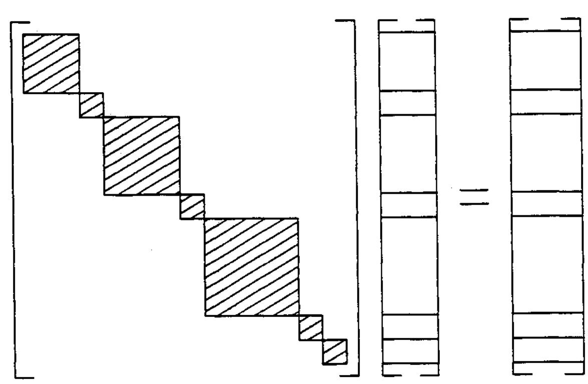

**Fig. 3b Structure of the dynamical equations for the system in Fig. 3a.**

and concomitantly, Eq. (13) reduces to

$$\begin{Bmatrix} a_0^{Q_j} \\ \alpha_0^j \end{Bmatrix} = R^j \begin{Bmatrix} \ddot{\tau}^j \\ \ddot{\theta}^j \end{Bmatrix} \tag{50}$$

permitting the computation of $\ddot{\eta}^j$ from Eq. (43). Once the dynamical equations for the base body are formed, the rest of the dynamical equations are obtained by using the kinematical equations "going up the tree." The algorithm is block diagonal because it breaks down the dynamical equation of the $n$-degree-of-freedom system into sub-blocks for each body $j$, each sub-block requiring inversion of matrices of order $M_j$ and $(T_j + R_j)$. The structure of the dynamical equations to be solved for the highest derivatives is schematically shown in Fig. 3b for a system with one rigid body and three articulated flexible bodies of Fig. 3a. The overall algorithm is summarized next.

## Summary of the Algorithm for a Tree Configuration

First forward pass:

Step 1) For $j = 1, \ldots, N$ (number of bodies) compute $\omega^j$ from Eq. (9), $a_t^{Q_j}$ from Eq. (15), $\alpha_t^j$ from Eq. (14), $E^j$ (a constant matrix), $A^j$ from Eq. (21), $Y_1^j$ from Eq. (22), $M_1^j$ from Eq. (24), $M_2^j$ from Eq. (25), $X^j$ from Eq. (26), $W^j$ from Eq. (34), $N^j$ from Eq. (35); and specify $S_j^{c(j)}$ in Eq. (41).

Backward pass:

Step 2) For $j = N, \ldots, 1$ compute $M_3^j$ and $Y_2^j$ from Eq. (27), $\nu^j$ from Eq. (30), and $M$, and $X$ from Eq. (31). If $j = 1$, go to step 5.

Step 3) Compute the updates defined in Eqs. (44) and (47).

Step 4) Replace $j$ by $j-1$. Go to step 2.

Second forward pass:

Step 5) For $j = 1$, compute in sequence Eqs. (49), (50), and (43). Set $j = 2$.

Step 6) Compute in sequence Eqs. (33), (29), (13), and (43).

Step 7) If $j = n$, stop. Otherwise, replace $j$ by $j+1$ and go to step 6.

## Treatment of Constraints

This section is concerned with systems of bodies subjected to motion constraints and motion-induced stiffness. When there are constraints on the system such as that represented by a closed loop of bodies, one approach[28,29] that lends itself to the block-diagonal formulation is to cut the loop, impose equal and opposite constraint forces and torques on the bodies separated, and treat the resulting system as one of a tree configuration. Figures 4a and 4b illustrate the situation, where the motion as well as the constraint forces and torques are unknown. Let $F_c$ and $T_c$ denote the body measure numbers of the constraint force and torque acting on one of the bodies at the joint that is cut, and introduce the notation

$$\lambda = \begin{Bmatrix} F_c \\ T_c \end{Bmatrix} \tag{51}$$

and define for the same "terminal" body $j$

$$H_1^j = \phi_c^{j^T} - \sum_{i=1}^{6} G_{12+i}^j \eta^j S_i^j$$

$$H_2^j = \begin{bmatrix} I & 0 \\ \tilde{r}_c^j & I \end{bmatrix} \tag{52}$$

where $G_{12+i}^j$ refers to the $i$th generalized geometric stiffness due to interbody force and moment; and $\phi_c$ and $r_c$ refer to the modal displacement at, and position vector from $Q_j$ to, the cut joint, respectively. For the other terminal body at the cut joint, $\phi_c$ and $r_c$ are used with the signs reversed and with appropriate basis transformation. Now with this additional force and torque of constraint, Eqs. (20) and (23) become, respectively,

$$E^j \ddot{\eta}^j = A^j \begin{Bmatrix} a_0^{Q_j} \\ \alpha_0^j \end{Bmatrix} + Y_1^j + H_1^j \lambda \tag{53}$$

$$\begin{Bmatrix} f^{*j} - f_e^j \\ t^{*j} - t_e^j \end{Bmatrix} = M_1^j \begin{Bmatrix} a_0^{Q_j} \\ \alpha_0^j \end{Bmatrix} + M_2^j \ddot{\eta}^j + X^j - H_2^j \lambda \tag{54}$$

Use of Eq. (53) in Eq. (54) leads to the terms defined in Eq. (27) as before, and the new term

$$H_3^j = H_2^j - M_2^j E^{j^{-1}} H_1^j \tag{55}$$

Equations for the hinge degrees of freedom are modified from Eq. (29) to

$$\begin{Bmatrix} \ddot{\tau}^j \\ \ddot{\theta}^j \end{Bmatrix} = -\nu^{j^{-1}} R^{j^T} \left[ M_3^j \begin{Bmatrix} \hat{a}_0^{Q_j} \\ \hat{\alpha}_0^j \end{Bmatrix} + Y_2^j - H_3^j \lambda \right] + \nu^{j^{-1}} \begin{pmatrix} f_h^j \\ t_h^j \end{pmatrix} \tag{56}$$

With this the expression for the equilibrated set of forces and torques at the hinge, $Q_j$ is obtained as

$$\begin{Bmatrix} f^{*j} - f_e^j \\ t^{*j} - t_e^j \end{Bmatrix} = M \begin{Bmatrix} \hat{a}_0^{Q_j} \\ \hat{\alpha}_0^j \end{Bmatrix} + X - H\lambda \tag{57}$$

where the definitions of $M$ and $X$ are retained as in Eq. (31) and

$$H = H_3^j - M_3^j R^j \nu^{j^{-1}} R^{j^T} H_3^j \tag{58}$$

The equations corresponding to the modal coordinates for the inboard body now become

$$E^{c(j)} \ddot{\eta}^{c(j)} = A^{c(j)} \begin{Bmatrix} a_0^{Q_{c(j)}} \\ \alpha_0^{c(j)} \end{Bmatrix} + Y_1^{c(j)} + H_1^{c(j)} \lambda \tag{59}$$

where the definitions of Eq. (44) remain unchanged, and geometric stiffness due to interbody forces on body $c(j)$ are accounted in

$$H_1^{c(j)} = \left[ \sum_{i=1}^{6} G_{12+i}^{c(j)} \eta^{c(j)} S_i^{c(j)} d_i^{c(j)} - N^{j^T} \right] H \tag{60}$$

Finally, shifting the forces and moments from $Q_j$ to $Q_{c(j)}$, using the updates of Eq. (47), defining further

$$H_3^{c(j)} = H_2^{c(j)} + W^{j^T} H \tag{61}$$

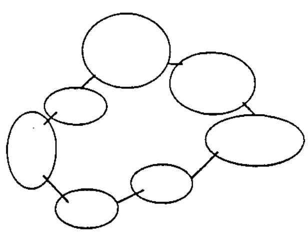

Fig. 4a System of bodies in a closed kinematic loop.

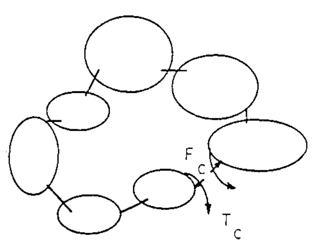


Fig. 4b Equivalent system with the loop cut at a hinge.

and using the procedure given earlier for deriving equations for hinge degrees of freedom yield

$$\begin{Bmatrix} \ddot{\tau}^{c(j)} \\ \ddot{\theta}^{c(j)} \end{Bmatrix} = -\nu^{c(j)^{-1}} R^{c(j)^T} \left[ M_3^{c(j)} \begin{Bmatrix} \hat{a}_0^{Q^{c(j)}} \\ \hat{\alpha}_0^{c(j)} \end{Bmatrix} + Y_2^{c(j)} - H_3^{c(j)} \lambda \right] + \nu^{c(j)^{-1}} \begin{pmatrix} f_h^{c(j)} \\ t_h^{c(j)} \end{pmatrix} \tag{62}$$

In summary, note that Eqs. (62) and (59) differ from Eqs. (48) and (43), respectively, only by the one term associated with the constraint forces and torques. In other words, setting the constraint forces and moments to zero recovers the equations for the tree configuration, as it must. To evaluate the second derivatives in Eqs. (62) and (59), one needs to know the unknown constraint forces and moments which show up in every dynamical equation. The solution is obtained by the additional consideration of the kinematical equations describing the constraint condition such as loop closure. However, a direct substitution of the constraint equations will destroy the block-diagonal feature of the dynamical equations. The following algorithm preserves the desired block-diagonal character of the coefficient matrix of the dynamical equations. Define for $j = 1$

$$f_1^1 = -\nu^{1^{-1}} R^{1^T} Y_2^1 + \nu^{1^{-1}} \begin{pmatrix} f_h^1 \\ t_h^1 \end{pmatrix}$$
$$c_1^1 = \nu^{1^{-1}} R^{1^T} H_3^1 \tag{63}$$

thus starting the sequence with

$$\begin{Bmatrix} \ddot{\tau}^1 \\ \ddot{\theta}^1 \end{Bmatrix} = f_1^1 + c_1^1 \lambda \tag{64}$$

Now, use of Eq. (13), where the zeroth body is the inertial frame, yields

$$\begin{Bmatrix} a_0^{Q_1} \\ \alpha_0^1 \end{Bmatrix} = g^1 + h^1 \lambda \tag{65}$$

where

$$g^1 = R^1 f_1^1$$
$$h^1 = R^1 c_1^1 \tag{66}$$

Reference to Eq. (53) and use of the definitions, again for $j = 1$,

$$f_2^1 = E^{1^{-1}} [A^1 g^1 + Y_1^1]$$
$$c_2^1 = E^{1^{-1}} [A^1 h^1 + H_1^1] \tag{67}$$

yield the equations for the modal coordinates of body 1 in the form

$$\ddot{\eta}^1 = f_2^1 + c_2^1 \lambda \tag{68}$$

Use of Eqs. (65) and (68) and the definitions

$$e^j = W^j g^{c(j)} + N^j f_2^{c(j)}$$
$$n^j = W^j h^{c(j)} + N^j c_2^{c(j)} \tag{69}$$

in Eq. (33) lead to the equations

$$\begin{Bmatrix} \hat{a}_0^{Q_j} \\ \hat{\alpha}_0^j \end{Bmatrix} = e^j + n^j \lambda \tag{70}$$

When this is substituted in Eq. (56), and the following generalized notations are introduced,

$$f_1^j = -\nu^{j^{-1}} R^{j^T} \left[ M_3^j e^j + Y_2^j \right] + \nu^{j^{-1}} \begin{pmatrix} f_h^j \\ t_h^j \end{pmatrix}$$
$$c_1^j = -\nu^{j^{-1}} R^{j^T} [M_3^j n^j - H_3^j] \tag{71}$$

the equations for the hinge degrees of freedom are obtained.

$$\begin{Bmatrix} \ddot{\tau}^j \\ \ddot{\theta}^j \end{Bmatrix} = f_1^j + c_1^j \lambda \tag{72}$$

Use of Eqs. (70) and (72) in Eq. (13) and the definitions

$$g^j = e^j + R^j f_1^j$$
$$h^j = n^j + R^j c_1^j \tag{73}$$

yield the general relation

$$\begin{Bmatrix} a_0^{Q_j} \\ \alpha_0^j \end{Bmatrix} = g^j + h^j \lambda \tag{74}$$

Finally, substitution of Eq. (74) in Eq. (53), and defining

$$f_2^j = E^{j^{-1}} [A^j g^j + Y_1^j]$$
$$c_2^j = E^{j^{-1}} [A^j h^j + H_1^j] \tag{75}$$

produce the equations for the modal coordinate for the $j$th body.

$$\ddot{\eta}^j = f_2^j + c_2^j \lambda \tag{76}$$

This completes the recursive generation of the differential equations of motion for a constrained system of flexible bodies. These equations will have to be solved together with the constraint equations. For holonomic constraints, the constraint conditions can be written as $m$ nonlinear algebraic equations in the $n$ generalized coordinates, $m < n$.

$$f(q_1, \ldots, q_n) = 0 \tag{77}$$

One way[6] of solving for the highest derivatives in the motion variables is to differentiate Eqs. (77) twice with respect to time and solve the resulting equations together with the dynamical equations. However, this approach is known[34] to give rise to growing drift in constraint satisfaction at the position/orientation level in the course of numerical integration. Recent mathematical research[35] has suggested that "any sound numerical method for solving the equations of motion in the constrained form must be stabilized by insuring that the position constraints are satisfied." A procedure that simultaneously satisfies the constraints at the position, velocity, and acceleration levels is given next.

Two differentiations of Eq. (77) with respect to time yield

$$J_i \dot{U}_i + J_d \dot{U}_d = -\{ \dot{J}_i U_i + \dot{J}_d U_d \} \tag{78}$$

where the following notations have been used:

$$J_i = \frac{\partial f}{\partial q_i}; \qquad J_d = \frac{\partial f}{\partial q_d}; \qquad U_i = \dot{q}_i; \qquad U_d = \dot{q}_d \tag{79}$$

with the set of generalized coordinates partitioned into a subset $q_i$ of independent generalized coordinates and the subset $q_d$ of dependent generalized coordinates. Now the dynamical equations generated sequentially in Eqs. (72) and (76) for bodies $j = 1, \ldots, N$ can be stacked together and written as follows:

$$\dot{U}_i = F_i + C_i \lambda \qquad \dot{U}_d = F_d + C_d \lambda \tag{80}$$

Substitution of Eqs. (80) in Eq. (78) provides the equations for the measure numbers of the constraint forces and torques.

$$[J_i C_i + J_d C_d]\lambda = -\{J_i F_i + J_d F_d + \dot{J}_i U_i + \dot{J}_d U_d\} \tag{81}$$

The dynamical differential equations are now explicitly known by using Eq. (81) in Eqs. (80). In situations where the coefficient matrix in Eq. (81) is singular, the penalty method of Ref. 36 may be used. To summarize, the following set of differential-algebraic equations[37] has to be solved for constrained dynamical systems.

$$\dot{U}_i = F_i - C_i[J_i C_i + J_d C_d]^{-1}\{J_i F_i + J_d F_d + \dot{J}_i U_i + \dot{J}_d U_d\} \tag{82}$$

$$\dot{q}_i = U_i \tag{83}$$

$$J_i U_i + J_d U_d = 0 \tag{84}$$

$$f(q_i, q_d) = 0 \tag{85}$$

## Computational Efficiency

A general purpose multibody dynamics code based on the formulation given here has been developed at the author's institution. In one early application to a wrap-rib antenna undergoing large deformation during deployment, modeled with 160 rigid segments attached to an L-shaped cantilever beam, with complex interbody forces and bodies connected to one another at two-degree-of-freedom hinges by nonlinear springs, the computational savings of the block-diagonal formulation over a coupled formulation has been significant (*26 times faster*). Computational efficiency of the algorithm can be augmented by symbolic code generation for specific problems.

## Conclusion

A comprehensive and computationally efficient formulation has been given for the equations of large overall motion of a system of flexible bodies subjected to motion-induced stiffness and constraints. For an arbitrary elastic body, the method requires a one-time computation of geometric stiffness matrices due to 12 distributed inertia loadings and six point loadings per hinge connection. By accounting for geometric stiffness due to inertia forces as well as interbody forces, the formulation retains its validity over a wide range of rotation and translation rates. For systems with configuration constraints, the formulation given in the paper satisfies the constraint condition at the position, velocity, and acceleration levels. The algorithm decouples the dynamical equations of each body within the system, and these latter equations are further decoupled into subsets requiring inversion of two matrices, one of order equal to the number of modes kept and the other of order equal to the number of relative rigid-body degrees of freedom for a body with respect to its inboard body. For constrained systems, an additional matrix of order equal to the number of constraints has to be inverted. The overall block-diagonal nature of the algorithm makes it computationally attractive, especially for systems where there are a large number of flexible bodies.

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## References

[1]Hooker, W. W., and Margulies, G., "The Dynamical Attitude Equations for an $n$-Body Satellite," *Journal of the Astronautical Sciences,* Vol. 7, No. 4, 1965, pp. 123–128.

[2]Kane, T. R., and Levinson, D. A., *Dynamics: Theory and Applications,* McGraw-Hill, New York, 1985, pp. 45, 159.

[3]Roberson, R. E., and Schwertassek, R., *Dynamics of Multibody Systems,* Springer-Verlag, 1988, Chap. 12.

[4]Rosenthal, D. E., and Sherman, M. A., "High Performance Multibody Simulations via Symbolic Equation Manipulation and Kane's Method," *Journal of the Astronautical Sciences,* Vol. 34, No. 3, 1986, pp. 223–239.

[5]Levinson, D. A., and Kane, T. R., "AUTOLEV—A New Approach to Multibody Dynamics," *Multibody Systems Handbook,* edited by W. Schiehlen, Springer-Verlag, Berlin, 1990, pp. 81–102.

[6]Bodley, C. S., Devers, A. D., Park, A. C., and Frisch, H. P., "A Digital Computer Program for the Dynamic Interaction Simulation of Controls and Structures (DISCOS)," Vols. I and II, NASA TP-1219, May 1978.

[7]Singh, R. P., van der Voort, R. J., and Likins, P. W., "Dynamics of Flexible Bodies in Tree Topology—A Computer Oriented Approach," *Journal of Guidance, Control, and Dynamics,* Vol. 8, No. 5, 1985, pp. 584–590.

[8]Kane, T. R., Ryan, R. R., and Banerjee, A. K., "Dynamics of a Cantilever Beam Attached to a Moving Base," *Journal of Guidance, Control, and Dynamics,* Vol. 10, No. 2, 1987, pp. 139–151.

[9]Eke, F. O., and Laskin, R. A., "On the Inadequacies of Current Multi-Flexible Body Simulation Codes," AIAA Guidance, Navigation and Control Conf., AIAA Paper 87-2248, Monterey, CA, Aug. 17–19, 1987.

[10]Banerjee, A. K., and Kane, T. R., "Dynamics of a Plate in Large Overall Motion," *Journal of Applied Mechanics,* Vol. 56, Dec. 1989, pp. 887–892.

[11]Padilla, C. E., and von Flotow, A. H., "Nonlinear Strain-Displacement Relations and Flexible Multibody Dynamics," *Journal of Guidance, Control, and Dynamics,* Vol. 15, No. 1, 1992, pp. 128–136.

[12]Likins, P. W., "Geometric Stiffness Characteristics of a Rotating Elastic Appendage," *International Journal of Solids and Structures,* Vol. 10, No. 2, 1974, pp. 161–167.

[13]Turcic, D. A., and Midha, A., "Generalized Equations of Motion for the Dynamic Analysis of Elastic Mechanism Systems," *ASME Journal of Dynamic Systems, Measurement, and Control,* Vol. 106, Dec. 1984, pp. 243–248.

[14]Modi, V. J., and Ibrahim, A. M., "On the Dynamics of Flexible Orbiting Structures," *Large Space Structures: Dynamics and Control,* edited by S. N. Atluri and A. K. Amos, Springer Verlag, New York, 1988, pp. 93–114.

[15]Ider, S. K., and Amirouche, F. M. L., "The Influence of Geometric Nonlinearities in the Dynamics of Flexible Tree-Like Structures," *Journal of Guidance, Control, and Dynamics,* Vol. 12, No. 6, 1989, pp. 830–837.

[16]Simo, J. C., and Vu-Quoc, L., "On the Dynamics of Flexible Bodies Under Large Overall Motions—The Plane Case: Parts I & II," *Journal of Applied Mechanics,* Vol. 53, Dec. 1986, pp. 849–863.

[17]Bakr, E. M., and Shabana, A. A., "Geometrically Nonlinear Analysis of Multibody Systems," *Computers and Structures,* Vol. 23, No. 6, 1986, pp. 739–751.

[18]Banerjee, A. K., and Dickens, J. M., "Dynamics of an Arbitrary Structure in Large Rotation and Translation," *Journal of Guidance, Control, and Dynamics,* Vol. 13, No. 2, 1990, pp. 221–227.

[19]Banerjee, A. K., and Lemak, M. E., "Multi-Flexible-Body Dynamics Capturing Motion-Induced Stiffness," *Journal of Applied Mechanics,* Vol. 58, Sept. 1991, pp. 766–775.

[20]Hollerbach, J. M., "A Recursive Lagrangian Formulation of Manipulator Dynamics and a Comparative Study of Dynamics Formulation Complexity," *IEEE Transactions on Systems, Man, and Cybernetics,* Vol. SMC-10, No. 11, 1980, pp. 730–736.

[21]Rosenthal, D. E., "An Order-$N$ Formulation for Robotic Systems," *Journal of Astronautical Sciences,* Vol. 38, No. 4, 1990, pp. 511–530.

[22]Anderson, K. S., "Recursive Derivation of Explicit Equations of Motion for Efficient Dynamic/Control Simulation of Large Multibody Systems," Ph.D. Dissertation, Stanford Univ., Stanford, CA, Sept. 1990.

[23]Rodriguez, G., "Spatially Recursive Filtering and Smoothing Methods for Multibody Dynamics," *Proceedings of the SDIO/NASA*

*Workshop on Multibody Simulations,* Jet Propulsion Lab., Pasadena, CA, Sept. 1987, pp. 1094–1121.

[24]Jain, A., "Unified Formulation of Dynamics for Serial Rigid Multibody Systems," *Journal of Guidance, Control, and Dynamics,* Vol. 14, No. 3, 1991, pp. 531–542.

[25]Bae, D. S., and Haug, E. J., "A Recursive Formulation for Constrained Mechanical System Dynamics: Part I, Open Loop Systems," *Mechanics of Structures and Machines,* Vol. 15, No. 3, 1987, pp. 359–382.

[26]Bae, D. S., and Haug, E. J., "A Recursive Formulation for Constrained Mechanical System Dynamics: Part II, Closed Loop Systems," *Mechanics of Structures and Machines,* Vol. 15, No. 4, 1987–88, pp. 481–506.

[27]Featherstone, R., "The Calculation of Robot Dynamics Using Articulated-Body Inertias," *International Journal of Robotics Research,* Vol. 2, No. 1, 1983, pp. 13–30.

[28]Keat, J. E., "Multibody System Order *n* Dynamics Formulation Based on Velocity Transform Method," *Journal of Guidance, Control, and Dynamics,* Vol. 13, No. 2, 1990, pp. 207–212.

[29]Chun, H. M., Turner, J. D., and Frisch, H. P., "A Recursive Order-*n* Formulation for DISCOS with Topological Loops and Intermittent Surface Contact," AAS/AIAA Astrodynamics Specialist Conf., Paper AAS 91-455, Durango, CO, Aug. 19–22, 1991.

[30]Anon., *Multi-Body Space Station Dynamics,* SSSIM Rev 2.0, Theory Manual, Dynacs Engineering Co., Clearwater, FL, April 6, 1990.

[31]Banerjee, A. K., "Order-*n* Formulation of Extrusion of a Beam with Large Bending and Base Rotation," *Journal of Guidance, Control, and Dynamics,* Vol. 15, No. 1, 1992, pp. 121–127.

[32]Padilla, C. E., and von Flotow, A. H., "Further Approximations in Flexible Multibody Dynamics," AIAA/ASME/ASCE/AHS/ASC 32nd Structures, Structural Dynamics, and Materials Conf., AIAA Paper 91-1115, Baltimore, MD, April 1991.

[33]Przemieniecki, J. S., *Theory of Matrix Structural Analysis,* McGraw-Hill, New York, 1968, pp. 386–391.

[34]Baumgarte, J., "Stabilization of Constraint and Integrals of Motion in Dynamical Systems," *Computer Methods in Applied Mechanics and Engineering,* Vol. 1, 1972, pp. 1–16.

[35]Fuhrer, C., and Leimkuhler, B., "Formulation and Numerical Solution of the Equations of Constrained Mechanical Motion," DFVLR Tech. Rept., DFVLR-FB 89-08, Oberpfaffenhofen, Germany, March 1989.

[36]Park, K. C., and Chiou, J. C., "Stabilization of Computational Procedures for Constrained Dynamical Systems," *Journal of Guidance, Control, and Dynamics,* Vol. 11, No. 4, 1988, pp. 365–370.

[37]Brenan, K. E., Campbell, S. L., and Petzold, L. R., *Numerical Solution of Initial Value Problems in Differential-Algebraic Equations,* Elsevier, New York, 1989.